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PLASMA DIFFUSION ACROSS A MAGNETIC FIELD

Philip ROSENAU* and James M. HYMAN
Center for Nonlinear Studies, Los Alamos National Laboratory, Theoretical Division, MS B284, Los Alamos, NM 87545, USA

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Using a simple model of a slowly diffusing plasma across a strong magnetic field, it is demonstrated that plasma mass and energy evolves from an initially given density and temperature distribution into isothermal state with a self-similar diffusion profile that depends only on its initial mass and energy.

1. Introduction and statement of the problem

In this letter we study the asymptotic behavior of a plasma slowly diffusing across a strong magnetic field [1–2]. Without confining walls, or walls that are adequately remote, an initially generated plasma with finite support is free to diffuse into the vacuum. The evolution of the solution of the prototype equations for the diffusion of mass and energy across the magnetic field is dominated by a diagonal diffusion tensor. In past studies, the decoupled problems for the diffusion of particles in an (essentially) isothermal plasma [3–4] or the diffusion of heat in a stationary plasma [5–8] have been analyzed. The present study addresses the more complex situation where both processes are coupled [8].

The equations of motion we will study are

$$\rho_t = (D_1\rho_x)_x, \tag{1a}$$

$$P_t = (\rho D_2 T_x)_x + (T D_1 \rho_x)_x, \quad x \in (-\infty, \infty), \tag{1b}$$

where $D_i = d_{0i}\rho^{\alpha_i}T^{\beta_i}$, $i = 1$ or 2, P is the plasma pressure, ρ is the density and T is the temperature. The ionic and electron temperatures are assumed to be equal and $P = \rho T$. The choice of the α's and β's depends on the details of the particular physical process being modelled. The initial data is specified over a bounded domain

$$\rho(x,0) = \rho_0(x), \quad P(x,0) = P_0(x), \quad x \in (-x_0, +x_0). \tag{2}$$

The divergence form of eqs. (1) guarantee that no additional energy or mass is added (or subtracted) after the process is initialized.

2. Analysis

Eq. (1) is an idealized model in slab geometry, where energy, mass and radiation transport are neglected. Yet, even in this physically idealized system, regularity conditions that guarantee the existence of smooth solutions are not known. In this paper we extend previous results on a related system of initial boundary value problems to understand the (dynamics) of the prototype equations (1). There is often a large gap between what constitutes a reasonable physical model and one amenable to mathematical analysis. In this spirit the presented model is intended to serve as a building block toward our understanding of plasma transport rather than its immediate applicability to a given plasma situation.

*Permanent address: Department of Mechanical Engineering, Technion, Haifa 3200, Israel.

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To present our main result, we first construct a selfsimilar solution of eqs. (1). For the initial conditions

$$\rho_0(x) = M_0\delta(x), \quad P_0(x) = E_0\delta(x), \tag{3}$$

where

$$M_0 = \int_{-x_0}^{x_0} \rho_0(x)\,\mathrm{d}x, \quad E_0 = \int_{-x_0}^{x_0} P_0(x)\,\mathrm{d}x, \tag{4}$$

and $\delta(x)$ is the Dirac measure. The appropriate solution satisfies

$$\begin{aligned} P(x,t) &= \rho(x,t)E_0/M_0, \\ \rho(x,t) &= f(\zeta)/t^{1/(2+\alpha_1)}, \end{aligned} \tag{5}$$

where

$$\zeta = x/\left[\left(M_0^{\alpha_1}t\right)^{1/(\alpha_1+2)}\right] \tag{6}$$

and

$$f(\zeta) = \left[\alpha_1\left(\zeta_f^2 - \zeta^2\right)/(2\alpha_1+4)\right]^{1/\alpha_1}, \tag{7}$$

if $\zeta \le \zeta_f$ and $f(\zeta) = 0$ otherwise.

Note that the position of the diffusing front ζ_f depends only on the total mass M_0 of the system and α_1. It follows from (5) that the self-similar solutions represent an isothermally diffusing plasma.

Out of the many group invariant solutions, the one presented has been selected out because of its key role in the late-time description of problem (1). That is, the leading term in the asymptotic behavior of a solution with an initial mass M_0 and energy E_0 is given by (5). Thus, its far-field behavior is almost independent of the structure in the initial data. This behavior is a natural generalization of single equation case [9].

Since we have not yet obtained a rigorous proof of the attractive nature of the self-similar solution, we performed a series of numerical experiments to confirm this property

The isothermal nature of the solution dominates so strongly that the *specific form of the second diffusion coefficient is of little importance*. A typical transit to the self-similar regime which occurs quite quickly is shown in fig. 1. After the initial transients, the temperature equilibrates and eq. (1b) merely duplicates eq. (1a). Hence, the solution dynamics are almost identical to the single diffusion equation case.

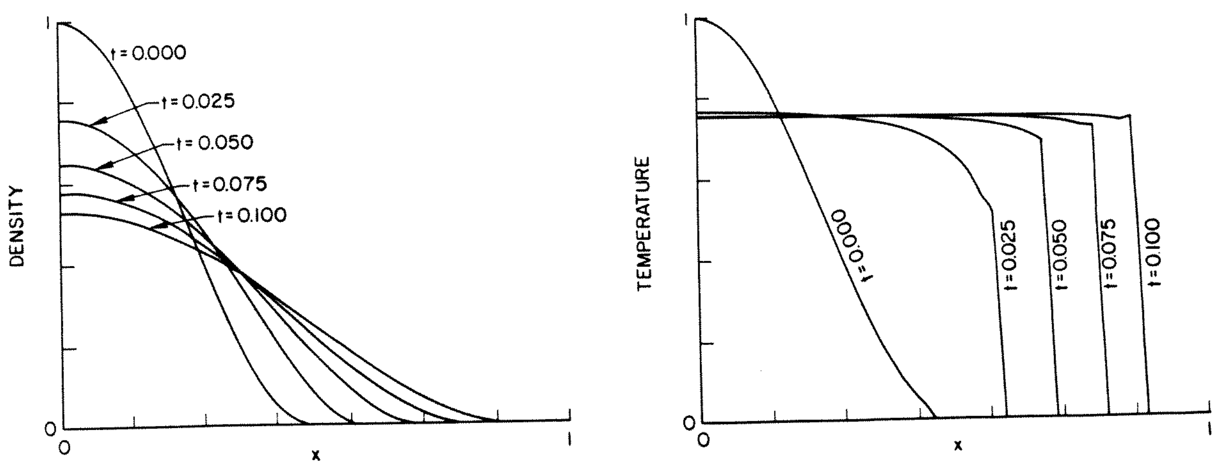

Fig. 1. The initial transient solution for $\alpha_1 = \frac{1}{2}$, $\beta_1 = \frac{1}{2}$, $\alpha_2 = 1$, $\beta_2 = 1$, $d_{01} = 1$, and $d_{02} = 5$.

3. Closing remarks

For a single equation it has been shown [9] that if a finite mass M_0 is distributed over the whole space, then the thermal diffusion as given by

$$\rho(x)T_t = [A(T)]_{xx}$$

leads to the isothermalization of the medium if A satisfies $A(0) = 0$, $A'(0) \geq 0$ and $A'(T) > 0$ for $T > 0$. That is,

$$T(x, t) \to T_a \equiv \int_{-\infty}^{\infty} \rho(x) T(x,0)\, dx / M_0.$$

As might be anticipated on the basis of physical considerations, the diffusion of heat in a finite mass medium results in isothermalization of the medium irrespective of how the mass is distributed.

Currently, we are studying the impact of radiation on the dynamics of the system. In this case, energy is not conserved. Also, it was recently shown [10] that if an energy source is included (and density is fixed to be constant) the asymptotic selfsimilar regime is very different from the energy conserving case. Hence expect radiation to have an important impact on the plasma dynamics.

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